

# Moduli of flat connections in positive characteristic

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## Abstract

Exploiting the description of rings of differential operators as Azumaya algebras on cotangent bundles, we show that the moduli stack of flat connections on a curve defined over an algebraically closed field of positive characteristic is étale locally equivalent to a moduli stack of Higgs bundles over the Hitchin base. We then study the interplay with stability and generalize a result of Laszlo–Pauly, concerning properness of the Hitchin map. Using Arinkin’s autoduality of compactified Jacobians we extend the main result of Bezrukavnikov–Braverman, the Langlands correspondence for  $D$ -modules in positive characteristic for smooth spectral curves, to the locus of integral spectral curves.

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## 1 Introduction

Let  $X$  be a smooth proper curve defined over an algebraically closed field  $k$ , and  $n$  a positive integer. We will be concerned with the study of various moduli stacks or spaces and their relation with each other. We use the notation  $\mathcal{M}_{\text{Dol}}$  for the moduli stack of rank  $n$  Higgs bundles. Recall that a Higgs bundle is a pair  $(E, \theta)$  consisting of a vector bundle  $E$  on  $X$  and a *Higgs field*

$$\theta : E \rightarrow E \otimes \Omega_X^1.$$

The moduli stack  $\mathcal{M}_{\text{Dol}}$  is equipped with a morphism  $\chi_{\text{Dol}} : \mathcal{M}_{\text{Dol}} \rightarrow \mathcal{A}$ , where  $\mathcal{A}$  denotes the Hitchin base

$$\mathcal{A} = \bigoplus_{i=1}^n H^0(X, \Omega_X^{\otimes i}),$$

and  $\chi_{\text{Dol}}$  sends the Higgs pair  $(E, \theta)$  to the coefficients of the characteristic polynomial of  $\theta$ . The induced morphism for the moduli space of semistable Higgs bundles  $\mathcal{M}_{\text{Dol}}^{\text{ss}}$  is proper and its generic fibre is an abelian variety.

We denote by  $\mathcal{M}_{\text{dR}}$  the moduli stack of pairs  $(E, \nabla)$ , where  $E$  is a rank  $n$  vector bundle on  $X$  and

$$\nabla : E \rightarrow E \otimes \Omega_X^1$$

is an algebraic flat connection.<sup>1</sup>

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<sup>1</sup>We also use the term *local system* for a pair  $(E, \nabla)$ .

The aim of this paper is to study the moduli stack  $\mathcal{M}_{\mathrm{dR}}$  for curves defined over an algebraically closed field  $k$  of positive characteristic  $p > 0$ , which we fix once and for all. We will demonstrate that in this set-up the geometry of  $\mathcal{M}_{\mathrm{dR}}$  is very similar to the geometry of  $\mathcal{M}_{\mathrm{Dol}}$ .

Given a tangent vector field  $\partial$  on a smooth scheme, the  $p$ -th iterate  $\partial^p$  acts on functions as a derivation. The corresponding vector field is denoted by  $\partial^{[p]}$ . This allows us to define the  $p$ -curvature of a flat connection in positive characteristic. It is given by the twisted endomorphism  $E \rightarrow E \otimes \mathrm{Fr}^* \Omega_{X^{(1)}}^1$ <sup>2</sup> corresponding to

$$(\nabla_{\partial})^p - \nabla_{\partial^{[p]}}.$$

It has been shown in [LP01] that the characteristic polynomial of the  $p$ -curvature gives rise to a natural morphism

$$\chi_{\mathrm{dR}} : \mathcal{M}_{\mathrm{dR}}(X) \rightarrow \mathcal{A}^{(1)}$$

to the Hitchin base. We show that from the perspective of descent theory this map is indistinguishable from the Hitchin fibration  $\mathcal{M}_{\mathrm{Dol}}(X^{(1)}) \rightarrow \mathcal{A}^{(1)}$ .

**Theorem 1.1.** *The  $\mathcal{A}^{(1)}$ -stacks  $\mathcal{M}_{\mathrm{dR}}(X)$  and  $\mathcal{M}_{\mathrm{Dol}}(X^{(1)})$  are étale locally equivalent<sup>3</sup> over the Hitchin base  $\mathcal{A}^{(1)}$ . The same statement is true for the moduli spaces  $\mathcal{M}_{\mathrm{dR}}^{\mathrm{ss}}(X)$  and  $\mathcal{M}_{\mathrm{Dol}}^{\mathrm{ss}}(X^{(1)})$ .*

The proof of this Theorem relies on the powerful techniques introduced in [BMR] and [BB]. It has been shown that the ring of differential operators in positive characteristic can be described as an Azumaya algebra on the cotangent bundle. As Higgs bundles can be thought of as certain coherent sheaves on cotangent bundles, this allows us to interpret local systems as twisted versions of Higgs bundles. In [BB] it has been shown that this twistedness carries over to the moduli stacks, if we restrict to the open substacks corresponding to smooth spectral curves. Our main result 1.1 is a generalization of this observation to all spectral curves, including highly singular and non-reduced ones. As such, the ideas relevant to the proof are a natural continuation of those found in [BB].

After establishing Theorem 1.1 we give two applications. First we deduce properness of the Hitchin map introduced by Laszlo and Pauly ([LP01, Prop. 5.1]).

**Corollary 1.2.** *The morphism  $\chi_{\mathrm{dR}} : \mathcal{M}_{\mathrm{dR}}^{\mathrm{ss}} \rightarrow \mathcal{A}^{(1)}$  is proper.*

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<sup>2</sup>The upper index (1) denotes the Frobenius twist of a variety, the reader may ignore it for now.

<sup>3</sup>In this paper every covering in the étale topology is assumed to be fppf and étale.

Let us note that it is a consequence of the main result of [LP01] that  $\chi_{\mathrm{dR}}^{-1}(0)$  is proper. Therefore Corollary 1.2 extends this result from a single fibre to the entire fibration.

As a second application we extend the main result of [BB]. We establish a derived equivalence between the derived category of certain  $D$ -modules on the moduli stack of bundles  $\mathrm{Bun}(X)$  and the derived category of coherent sheaves on an open substack  $\mathcal{M}_{\mathrm{dR}}^{\mathrm{int}}(X) \subset \mathcal{M}_{\mathrm{dR}}$ . As expected, this equivalence respects the action of the Hecke and multiplication operators. Due to the limitations of what is currently known about autoduality of compactified Jacobians in positive characteristic, we need to assume that the characteristic of the base field satisfies the estimate  $p > 2n^2(h-1) + 1$ , where  $h$  denotes the genus of  $X$ . We use the notation  $\mathcal{M}_{\mathrm{dR}}^{\mathrm{int}}(X)$  and  $\mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}}(X^{(1)})$  for the moduli stacks of local systems respectively Higgs bundles of rank  $n$  with integral spectral curve.

**Corollary 1.3.** *There exists a canonical equivalence of derived categories intertwining the Hecke operator with the multiplication operator associated to the universal bundle  $\mathcal{E}$  on  $\mathcal{M}_{\mathrm{dR}}(X) \times X$*

$$D_{\mathrm{coh}}^b(\mathcal{M}_{\mathrm{dR}}^{\mathrm{int}}, \mathcal{O}) \cong D_{\mathrm{coh}}^b(\mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}}, \mathcal{D}_{\mathrm{Bun}}).$$

Arinkin's autoduality of compactified Jacobians (cf. [Ari10]) plays an essential role in our argument. We observe that this result still holds in large enough characteristic, and use it to deduce the existence of an integral kernel giving rise to the equivalence. We relate the Hecke eigenproperty of this kernel to an analogous property for Arinkin's Fourier-Mukai transform (Theorem 4.16), which has been stated in [Ari10, sect. 6.4].

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## 2 D-modules

In this section we recall foundational results on the ring of differential operators in positive characteristic. More detailed expositions can be found in [BB] and [BMR].

Let  $X$  be a smooth variety over a field  $k$  of characteristic  $p > 0$ . The *Frobenius twist*  $X^{(1)}$  of  $X$  is given by the following cartesian diagram

$$\begin{array}{ccc} X^{(1)} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{F} & \operatorname{Spec} k, \end{array}$$

where  $F : k \rightarrow k$  denotes the Frobenius map  $\lambda \mapsto \lambda^p$ . There exists a canonical  $k$ -linear morphism  $Fr_X : X \rightarrow X^{(1)}$ , which is referred to as *Frobenius morphism*. If  $X$  is the zero set of a polynomial

$$f(x_1, \dots, x_n) = \sum a_{i_1 \dots i_n} x^{i_1} \dots x^{i_n},$$

then  $X^{(1)}$  is the zero set of the polynomial

$$\sum a_{i_1 \dots i_n}^p x^{i_1} \dots x^{i_n},$$

and  $Fr_X$  is given by  $x \mapsto x^p$ .

The ring of (crystalline) differential operators  $D_X$  is defined to be the universal enveloping algebra  $U\Theta_X$  of the Lie algebroid of tangent vector fields  $\Theta_X$ . By definition  $Fr_X$  is an affine morphism, therefore the study of  $D_X$ -modules is equivalent to the study of  $Fr_* D_X$ -modules. Similar identifications are used for other affine morphisms. The Theorem below states that relative to its centre  $Fr_* D_X$  is an Azumaya algebra, i.e. is étale locally equivalent to the endomorphism algebra of a vector bundle. We refer the interested reader to chapter 4 in Milne's book [Mil80] for more details on Azumaya algebras.

**Theorem 2.1** ([BMR]). *The centre  $Z(Fr_* D_X)$  is canonically equivalent to the structure sheaf of the cotangent bundle  $\pi_* \mathcal{O}_{T^*X^{(1)}} = \operatorname{Sym}^\bullet T_{X^{(1)}}^*$ . Moreover relative to its centre, the ring  $Fr_* D_X$  is an Azumaya algebra  $\mathcal{D}_X$  of rank  $p^{2 \dim X}$ .*

The identification of the centre with the cotangent bundle is realized by the  $p$ -curvature. If  $\partial$  is a vector field on  $X$ , its  $p$ th-power  $\partial^p$  acts again as a derivation on functions, according to the Leibniz rule. We denote this vector field by  $\partial^{[p]}$ . The difference  $\partial^p - \partial^{[p]} \in D_X$  can be shown to be a central element (cf. [BMR, Lemma 1.3.1]). This expression is  $p$ -linear in  $\partial$  and hence gives rise to a morphism  $\pi_* \mathcal{O}_{T^*X^{(1)}} \rightarrow Z(Fr_* D_X)$ .

This Theorem implies the existence of an equivalence of categories

$$D_X - \operatorname{Mod} \cong \mathcal{D}_X - \operatorname{Mod}.$$

Note that objects in the latter category are no longer sheaves on  $X$  but on  $T^*X^{(1)}$ .

### 3 BNR correspondence for flat connections

A Higgs field  $\theta$  on a vector bundle  $E/X$  is a section of  $\underline{\text{End}}(E) \otimes \Omega_X^1$ . To  $\theta$  we associate its characteristic polynomial

$$a(\lambda) = \det(\lambda - \theta),$$

with coefficients

$$a_i \in H^0(X, \Omega_X^{\otimes n-i})$$

for  $i = 0, \dots, n-1$ . The corresponding spectral curve  $Y_a$  is the closed subscheme of  $T^*X$  defined by the equation

$$\lambda^n + a_{n-1}\lambda^{n-1} \cdots + a_0 = 0,$$

where  $\lambda$  denotes the tautological section of the pullback  $\pi^* \Omega_X^1$  of the canonical bundle to the cotangent space  $T^*X$ . The affine space of spectral curves is denoted by

$$\mathcal{A} = \bigoplus_{i=1}^n H^0(X, \Omega_X^{\otimes i}).$$

It parametrizes the universal family  $\phi : Y \rightarrow \mathcal{A}$  of spectral curves and in particular gives rise to a finite morphism

$$\pi : Y \rightarrow X \times \mathcal{A}.$$

Zariski locally on  $X$  we can trivialize the sheaf of tangent vector fields  $\Theta_X$  and see that

$$\pi_* \mathcal{O}_Y = \bigoplus_{i=0}^{n-1} p_1^* \Theta_X^{\otimes i}$$

is locally free. In particular we obtain that  $\pi$  and  $\phi$  are flat morphisms. Moreover using [EGAIII2, Prop. 7.8.4] we see that  $\phi$  is cohomologically flat in degree zero. We record this observation for later use.

**Lemma 3.1.** *The morphism  $\phi : Y \rightarrow \mathcal{A}$  is proper, flat and cohomologically flat in degree zero. Moreover its geometric fibres are locally planar curves.*

Given a quasi-coherent sheaf  $L$  on a spectral curve  $Y_a$  then its push-forward  $\pi_{a,*} L$  is naturally endowed with a Higgs field. The Higgs sheaf  $\pi_{a,*} L$  is a Higgs bundle if and only if  $L$  is coherent and of pure support. The last condition means that the support of every non-trivial subsheaf is of same dimension. This construction induces a natural bijection between Higgs bundles and coherent sheaves of pure support on spectral curves. It is usually referred to as the BNR correspondence (see [BNR89] and [Sim94, Lemma 6.8]).

**Theorem 3.2** (BNR correspondence). *There is a natural bijection between rank  $n$  Higgs bundles  $(E, \theta)$  with characteristic polynomial  $a = \chi(\theta)$  and coherent sheaves of pure support  $L$  on the spectral curve  $Y_a$ , such that  $\mathrm{rk} \pi_{a,*} L = n$ .*

Our immediate goal is to find a similar correspondence for flat connections. This will allow us to relate local systems to Higgs bundles and eventually to prove Theorem 1.1.

### 3.1 Stack of Splittings

Given a morphism of schemes  $\pi : Y \rightarrow X$  and an Azumaya algebra  $\mathcal{D}$  over  $Y$  we say that  $\mathcal{D}$  splits relatively over  $X$ , if there exists an étale covering  $\{U_i\}_{i \in I}$  of  $X$ , such that  $\mathcal{D}$  is split over every fibre product  $U_i \times_X Y$ . Equivalently we can associate to  $\mathcal{D}$  a global section of  $R^2 \pi_*^{et} \mathbb{G}_m$  and say that relative splitting occurs if and only if this characteristic section vanishes. Our main result in this subsection is a generalization of a classical Theorem: every Azumaya algebra defined over a smooth complete curve<sup>4</sup> splits. We will remove the smoothness assumption and further relativize this statement.

**Definition 3.3.** *A relative curve is a morphism of finite type  $k$ -schemes  $\pi : Y \rightarrow X$ , which is proper, flat, cohomologically flat in degree zero, and has geometric fibres of dimension one.*

**Theorem 3.4.** *Let  $\pi : Y \rightarrow X$  be a relative curve, and  $\mathcal{D}$  an Azumaya algebra on  $Y$ . Then every Azumaya algebra over  $Y$  splits relatively over  $X$ .*

We will need the following result, which is stated in [FK88], in a remark after Lemma I.5.2.

**Theorem 3.5.** *Let  $X$  be a proper Noetherian scheme over an algebraically closed field  $k$  of dimension  $\leq 1$ , then  $H_{et}^2(X, \mathbb{G}_m) = 0$ . In particular every Azumaya algebra defined over  $X$  splits.*

*Proof.* Let  $X$  be a scheme and  $\mathcal{I}$  a quasi-coherent sheaf of ideals satisfying  $\mathcal{I}^2 = 0$ . We consider the closed immersion  $j : Y \rightarrow X$  given by this sheaf of ideals in  $\mathcal{O}_X$  and study the truncated exponential sequence

$$0 \longrightarrow \mathcal{I} \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow j_* \mathcal{O}_Y^\times \longrightarrow 1.$$

The exponential function  $\exp : \mathcal{I} \rightarrow \mathcal{O}_X^\times$  is defined by the expression  $f \mapsto 1 + f$  and satisfies  $\exp(f + g) = \exp(f) \exp(g)$ ,  $\exp(0) = 1$ . In particular the map

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<sup>4</sup>We emphasize that this statement is not true for non-algebraically closed fields.

takes values in the sheaf of abelian groups of units. The corresponding long exact sequence implies that  $H_{et}^2(X, \mathbb{G}_m) = H_{et}^2(Y, \mathbb{G}_m)$ , since  $H^i(X, \mathcal{I}) = 0$  for  $i > 1$ . As  $X$  is noetherian, we may assume by induction that  $X$  is reduced.

We denote now by  $j : Y \rightarrow X$  the normalization map. The morphism  $j$  is finite, therefore the functor  $j_*$  is exact (cf. [Mil80, Cor II.3.6]). We can now apply the étale cohomology functor to the short exact sequence

$$1 \rightarrow \mathcal{O}_X^\times \rightarrow j_* \mathcal{O}_Y^\times \rightarrow j_* \mathcal{O}_Y^\times / \mathcal{O}_X^\times \rightarrow 1$$

and obtain a long exact sequence

$$\cdots \rightarrow H_{et}^i(X, \mathcal{O}_X^\times) \rightarrow H_{et}^i(Y, \mathcal{O}_Y^\times) \rightarrow H^i(j_* \mathcal{O}_Y^\times / \mathcal{O}_X^\times) \rightarrow H_{et}^{i+1}(X, \mathcal{O}_X^\times) \rightarrow \cdots$$

The quotient sheaf  $j_* \mathcal{O}_Y^\times / \mathcal{O}_X^\times$  is supported at finitely many closed points, therefore all of its higher cohomology groups have to vanish. As a consequence we obtain that  $H_{et}^2(X, \mathcal{O}_X^\times) = 0$ .

As has been argued above, we may assume that  $X$  is smooth. The proof of this case is a classical application of a Tsen's theorem in Galois cohomology. More details can be found in [Mil80, Ex. 2.22 (d)].  $\square$

Recall that a splitting of an Azumaya algebra  $\mathcal{D}$  on  $X$  is a pair  $(\phi, E)$ , where  $E$  is a locally free sheaf on  $X$  and

$$\phi : \underline{\text{End}}(E) \rightarrow \mathcal{D}$$

is an isomorphism.

**Definition 3.6.** *Given an Azumaya algebra  $\mathcal{D}$  over  $Y$  and a morphism  $Y \rightarrow X$ , we define a lax 2-functor  $\mathcal{S}$  from the category  $\text{Sch}/X$  to the 2-category of groupoids sending a scheme  $U \rightarrow X$  to the groupoid of splittings of  $\mathcal{D}$  over  $U \times_X Y$ . Here a morphism between two splittings  $(\phi, E), (\psi, F)$  is defined to be a pair  $(\gamma, L)$  where  $L$  is a line bundle on  $U \times_X Y$  and  $\gamma : E \rightarrow F \otimes L$  is an isomorphism. This is a stack referred to as the stack of relative splittings of  $\mathcal{D}$  along  $\pi : Y \rightarrow X$ .*

To deduce the relative from the absolute case, we need to study the deformation theory of splittings.

**Lemma 3.7** (Commutation with inverse limits). *Let  $\pi : Y \rightarrow X$  be a relative curve and  $\mathcal{D}$  an Azumaya algebra over  $Y$ . Given a strict complete noetherian local ring  $\bar{A}$  together with a morphism  $\text{Spec } \bar{A} \rightarrow X$  it holds that  $\mathcal{D}$  splits over the base change  $\text{Spec } \bar{A} \times_X Y$ . In particular we obtain for the lax 2-functor  $\mathcal{S}$  that*

$$\mathcal{S}(\bar{A}) \rightarrow \text{inv-lim } \mathcal{S}(\bar{A}/\mathfrak{m}^n)$$

*is an isomorphism.*



*Proof.* In the course of the proof we will assume that  $X = \operatorname{Spec} \bar{A}$ . We denote the base change  $Y \times_X \operatorname{Spec} A/\mathfrak{m}_A^n$  by  $Y_n$ . The corresponding formal scheme is denoted by  $\hat{Y}$ .

Grothendieck's Existence Theorem states that the abelian category of coherent sheaves on  $Y$  is equivalent to the abelian category of coherent sheaves on the formal scheme  $\hat{Y}$ , which is a projective 2-limit category of the categories  $\operatorname{Coh}(Y_n)$  (cf. [Ill05, Thm 8.4.2]). According to Theorem 3.5 there exists a splitting  $S_n$  of  $\mathcal{D}|_{Y_n}$  for every  $n$ . In order to obtain a splitting of  $\mathcal{D}$  we need to choose compatible splittings  $S_n$ . This is always possible, as  $S_{n+1}|_{Y_n}$  differs from  $S_n$  by a line bundle. Since  $\pi : Y \rightarrow X$  is a relative curve, we can lift the difference line bundle from  $Y_n$  to  $Y_{n+1}$ . Therefore a sequence of compatible splittings  $S_n$  of  $\mathcal{D}_{Y_n}$  exists.

To prove the latter statement we use the existence of a splitting and reduce to the analogous statement for line bundles. This is an easy consequence of Grothendieck's Existence Theorem quoted above, since line bundles are characterized as invertible coherent sheaves.  $\square$

We refer the reader to [Art74] for the definition of the deformation theory of a stack. Note that in our case the deformation problem is unobstructed, as we are dealing with a relative curve.

**Lemma 3.8** (Deformation theory of Splittings). *For a relative curve  $\pi : Y \rightarrow X$  with an Azumaya algebra  $\mathcal{D}$  over  $Y$ . The lax 2-functor  $\mathcal{S}$  has the same deformation theory as  $\operatorname{Pic}$ .*

*Proof.* This follows directly from the fact that  $\mathcal{S}$  is a  $\operatorname{Pic}$  quasi-torsor. By this we mean that if a splitting exists, then it gives rise to an identification of the groupoid of splittings with the groupoid of line bundles. Moreover Lemma 3.7 shows that  $\mathcal{D}$  splits on formal fibres.  $\square$

According to a Theorem of M. Artin (cf. [Art74, Thm 5.3]), the deformation theory of a stack allows us to decide whether it is algebraic or not.

**Theorem 3.9.** *Let  $\pi : Y \rightarrow X$  be a relative curve and let  $\mathcal{D}$  be an Azumaya algebra over  $Y$ . The lax 2-functor  $\mathcal{S}$  is representable by an algebraic stack.*

*Proof.* It has been shown in [Art69] that  $\operatorname{Pic}$  is an algebraic stack. According to Lemma 3.8,  $\mathcal{S}$  and  $\operatorname{Pic}$  have the same deformation theory, thus both of them have to be algebraic.  $\square$

**Lemma 3.10.** *Let  $\pi : Y \rightarrow X$  be a relative curve and let  $\mathcal{D}$  be an Azumaya algebra over  $Y$ . Then the algebraic stack  $\mathcal{S}$  is smooth over  $X$ .*

*Proof.* This is a simple verification of the criterion for formal smoothness. We have already seen that there exists no obstruction to lifting splittings in the curve case in the proof of Lemma 3.7. Therefore the structural morphism  $\mathcal{S} \rightarrow X$  is smooth.  $\square$

*Proof of Theorem 3.4.* According to Lemma 3.10, the morphism of stacks  $\mathcal{S} \rightarrow X$  is smooth. In particular it has a section étale locally.  $\square$

**Lemma 3.11.** *The stack  $\mathcal{S}$  is étale locally isomorphic to  $\mathcal{P}\mathrm{ic}(Y/X)$ . In particular it is locally of finite presentation, smooth and universally open over the base.*

*Proof.* We have seen in Theorem 3.4 that every Azumaya algebra splits relatively over  $X$ . Therefore there exists an étale cover  $\{U_i\}$  of  $X$ , such that  $\mathcal{D}$  splits over  $U_i \times_X Y$ . Since two splittings of an Azumaya algebra are related by line bundles this gives rise to an isomorphism  $U_i \times_X \mathcal{S} \cong \mathcal{P}\mathrm{ic}(U_i \times_X Y/U_i)$ .

See Proposition 9.4.17 in [Kle05] for a proof that  $\mathcal{P}\mathrm{ic}(Y/X)$  is locally of finite presentation. Moreover  $\mathcal{P}\mathrm{ic}(Y/X)$  is smooth as the deformation theory of line bundles on curves is unobstructed. A flat morphism which is locally of finite presentation is universally open according to Proposition 2.4.6 in EGA IV.2 (cf. [EGAIV2]).  $\square$

## 3.2 Local systems

We can now establish an analogue of Theorem 3.2 for local systems. In the case of smooth spectral curves this description was contained in the proof of Lemma 4.8 in [BB]. We denote by  $\pi : Y \rightarrow X \times \mathcal{A}^{(1)}$  the universal spectral cover of  $X^{(1)}$  parametrized by  $\mathcal{A}^{(1)}$ .

**Proposition 3.12** (BNR for local systems). *Giving an  $S$ -family of local systems of rank  $n$  on  $X$  is equivalent to giving a morphism  $a : S \rightarrow \mathcal{A}^{(1)}$ , a coherent sheaf  $\mathcal{E}$  on  $Y \times_{\mathcal{A}^{(1)}} S$ , which is flat over  $S$ , carries a structure of a  $\mathcal{D}_X$ -module, is geometrically fibrewise of pure support and satisfies  $\mathrm{rk} \pi_* \mathcal{E} = pn$ .*

*Proof.* We use the equivalence of categories

$$D_X - \mathrm{Mod} \cong \mathcal{D}_X - \mathrm{Mod}.$$

Recall that under this equivalence, a local system  $(E, \nabla)$  is sent to the  $Fr_* \mathcal{D}_X$ -module  $Fr_* E$ , which gives rise to a  $\mathcal{D}_X$ -module  $\mathcal{E}$ .

The coherent sheaf  $Fr_* E$  is locally free of rank  $pn$ , the  $p$ -curvature of  $\nabla$  endows  $Fr_* E$  with a Higgs field. An  $S$ -family  $(E, \nabla)$  of local systems gives therefore rise to a morphism

$$b : S \rightarrow \mathcal{A}_{pn}^{(1)},$$

i.e. a characteristic polynomial of degree  $pn$ . According to the BNR correspondence for Higgs bundles (Theorem 3.2) the sheaf  $\mathcal{E}$  on  $S \times T^*X^{(1)}$  is geometrically fibrewise pure and supported on the spectral curve  $Y_b$  corresponding to the characteristic polynomial  $b$ .

In order to conclude the proof of this proposition, we need to show that  $b = a^p$  for a degree  $n$  characteristic polynomial  $a$ , and that  $\mathcal{E}$  is supported on the corresponding spectral curve.

According to Theorem 3.4 we may replace  $S$  by an étale covering and assume therefore that  $\mathcal{D}_X$  splits on the support of  $\mathcal{E}$ . Let  $x$  be a geometric point of  $X \times S$  and  $U = \text{Spec } \mathcal{O}$  the spectrum of the corresponding henselian ring. We consider the base change  $Y \times_X U = V \rightarrow U$ . Because it is a finite morphism, we conclude that  $V$  is the spectrum of a product of local algebras ([Mil80, Thm 4.2(b)]). By assumption we know that  $\mathcal{D}|_V$  is split. In particular there exists an isomorphism  $\mathcal{D}|_V \cong \underline{\text{End}}(M)$ , where  $M$  denotes a rank  $p$  vector bundle on  $V$ . Since  $\Gamma(\mathcal{O}_V)$  is a product of local rings,  $M$  is free. In particular we can identify  $\mathcal{D}|_V$  with the matrix algebra  $M_p(\mathcal{O}_V)$ .

This implies the existence of a coherent sheaf  $\mathcal{F}$ , s.t.

$$\mathcal{E} = \bigoplus_{i=1}^p \mathcal{F}.$$

We conclude that  $\mathcal{F}$  and  $\mathcal{E}$  are supported on the spectral curve  $Y_a$ .  $\square$

One consequence of the above discussion is the existence of a canonical morphism from the stack of flat connections  $\mathcal{M}_{\text{dR}}$  to the affine space  $\mathcal{A}^{(1)}$ . This analogue of the celebrated Hitchin map was studied by Y. Laszlo and C. Pauly in [LP01].

**Definition 3.13.** *The morphism  $\chi_{\text{dR}} : \mathcal{M}_{\text{dR}}(\Sigma) \rightarrow \mathcal{A}^{(1)}$*

$$(a : S \rightarrow \mathcal{A}^{(1)}, \mathcal{E}) \mapsto (a : S \rightarrow \mathcal{A}^{(1)})$$

*is called the twisted Hitchin morphism.*

Note that a direct definition of this morphism requires some work (see Proposition 3.2 in [LP01]), in fact the discussion above can be seen as a calculation free proof of this Proposition.

Moreover we can give the first application to the geometry of the stack of flat connections  $\mathcal{M}_{\text{dR}}$ .

**Corollary 3.14.** *Let  $\chi_{\text{dR}} : \mathcal{M}_{\text{dR}}(\Sigma^{(1)}) \rightarrow \mathcal{A}^{(1)}$  be the twisted Hitchin map derived from the  $p$ -curvature. We denote by  $\chi_{\text{Dol}} : \mathcal{M}_{\text{Dol}}(\Sigma^{(1)}) \rightarrow \mathcal{A}^{(1)}$  the usual Hitchin morphism. Then for  $a \in \mathcal{A}^{(1)}$ , we have that  $\chi_{\text{dR}}^{-1}(a)$  and  $\chi_{\text{Dol}}^{-1}(a)$  are non-canonically isomorphic.*

*Proof.* We have already noted that Azumaya algebras split on spectral curves (cf. Theorem 3.5). Every choice of a splitting  $S$  provides us with an equivalence of fibres. A  $\mathcal{D}$ -module  $\mathcal{F}$  on a spectral curve  $Y_a$  is sent to the  $\mathcal{O}_{Y_a}$ -module  $\underline{\mathrm{Hom}}_{\mathcal{D}}(S, \mathcal{F})$  and a  $\mathcal{O}_{Y_a}$ -module  $L$  is sent to  $L \otimes S$ . Both maps preserve the notion of purity, therefore we can apply the BNR correspondence as in Theorem 3.2 and Proposition 3.12, to conclude that this construction provides an isomorphism  $\chi_{\mathrm{dR}}^{-1}(a) \cong \chi_{\mathrm{Dol}}^{-1}(a)$ .  $\square$

### 3.3 Local equivalence of moduli stacks

In this subsection we use the BNR correspondence 3.12 and Corollary 3.4 to show that the moduli stack of Higgs bundles is étale locally equivalent to the moduli stack of local systems.

Let  $X$  be a complete smooth curve over an algebraically closed field  $k$  of characteristic  $p > 0$ . We denote by  $\chi : \mathcal{M}_{\mathrm{Dol}}(X^{(1)}) \rightarrow \mathcal{A}^{(1)}$  the Hitchin fibration mapping a Higgs bundle to the characteristic polynomial of its Higgs field, and  $\chi_{\mathrm{dR}} : \mathcal{M}_{\mathrm{dR}} \rightarrow \mathcal{A}^{(1)}$  the deformed Hitchin fibration mapping a local system to the characteristic polynomial of its  $p$ -curvature. These two morphisms induce the structure of an  $\mathcal{A}^{(1)}$ -stack on their domains.

**Theorem 3.15.** *There exists a canonical isomorphism*

$$\mathcal{S} \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\mathrm{dR}} \cong \mathcal{S} \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\mathrm{Dol}}.$$

*The  $\mathcal{A}^{(1)}$ -stacks  $\mathcal{M}_{\mathrm{dR}}(X)$  and  $\mathcal{M}_{\mathrm{Dol}}(X^{(1)})$  are étale locally equivalent, i.e. there exists an étale cover  $\{U_i\}_{i \in I}$  and isomorphisms of  $U_i$ -stacks*

$$U_i \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\mathrm{Dol}}(X^{(1)}) \cong U_i \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\mathrm{dR}}(X).$$

*Proof.* Given a splitting  $S$ , we send a  $\mathcal{D}$ -module  $F$  to  $\underline{\mathrm{Hom}}_{\mathcal{D}}(S, F)$  and a  $\mathcal{O}$ -module  $E$  to  $E \otimes S$ . Both functors are easily seen to be support decreasing, and since they are inverse to each other they must preserve support. Thus they respect the notion of purity. Proposition 3.12 and its analogue for Higgs bundles Theorem 3.2 finish the proof of the first assertion.

Lemma 3.1 and Theorem 3.4 imply that we can choose an étale cover  $\{U_i\}_{i \in I}$  and splittings  $S_i$  of  $\mathcal{D}_X$  on  $U_i \times_{\mathcal{A}^{(1)}} Y$ . We obtain isomorphisms

$$U_i \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\mathrm{Dol}}(X^{(1)}) \cong U_i \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\mathrm{dR}}(X).$$

$\square$

To illustrate this theory we may consider the example<sup>5</sup> of an elliptic curve  $X$ . The moduli space of flat line bundles on  $X$  is a group scheme. It arises

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<sup>5</sup>I thank Christian Pauly for explaining this example to me.

as an extension of  $\mathcal{P}\mathrm{ic}(X)$  by the one-dimensional vector space  $H^0(X, \Omega_X^1)$ . The Hitchin morphism  $\chi_{\mathrm{dR}} : \mathcal{M}_{\mathrm{Dol}} \rightarrow \mathcal{A}^{(1)}$  maps down to the one-dimensional vector space  $\mathcal{A}^{(1)} = H^0(X^{(1)}, \Omega_{X^{(1)}}^1)$ . In an attempt to construct a section of  $\chi_{\mathrm{dR}}$ , one has to study flat connection on the trivial line bundle  $(\mathcal{O}, d + \omega)$  with given  $p$ -curvature. This corresponds to finding a right-inverse to the map  $H^0(X, \Omega^1) \rightarrow \mathcal{M}_{\mathrm{dR}} \rightarrow H^0(X^{(1)}, \Omega_{X^{(1)}}^1)$ . According to formula 2.1.16 in [Ill79] this map is the sum of a  $p$ -linear and a linear map of vector spaces. Without loss of generality we may assume it is the map  $\lambda \mapsto \lambda^p - \lambda$ . We see that after base-change of  $\chi_{\mathrm{dR}}$  along the étale Artin-Schreier morphism, it is possible to trivialize  $\chi_{\mathrm{dR}}$ . This étale local section can be used to define an isomorphism

$$\mathcal{M}_{\mathrm{Dol}}(X^{(1)}) \times_{\mathcal{A}^{(1)}} \mathcal{A}^{(1)} \cong \mathcal{M}_{\mathrm{dR}}(X^{(1)}) \times_{\mathcal{A}^{(1)}} \mathcal{A}^{(1)},$$

where we base change along the Artin-Schreier morphism.

### 3.4 Stability

In this section we investigate the interaction of the local equivalence of moduli stacks in Theorem 3.15 with stability. Unmindful choice of a splitting in the proof of this Theorem will lead to the degree of the underlying bundles to be both scaled and shifted. If the spectral curves has several components this shift might differ between the components, which would certainly mean that stability is not preserved. Nonetheless, it is possible to single out a connected component  $\mathcal{S}^0 \subset \mathcal{S}$  of good splittings, where the degree of underlying bundles will only be scaled. First we demonstrate this phenomenon for smooth spectral curves.

**Lemma 3.16.** *Let  $a \in \mathcal{A}^{(1)}$  be the characteristic polynomial of a smooth spectral curve  $Y_a \rightarrow X^{(1)}$ . Given a splitting  $S$  of  $\mathcal{D}_X$  on  $Y_a$  the induced isomorphism of Hitchin fibres  $\chi_{\mathrm{Dol}}^{-1}(a) \rightarrow \chi_{\mathrm{dR}}^{-1}(a)$  sends a degree  $d$  Higgs bundle to a degree  $pd - (1-p)(1-h)n + \deg S$  local system, where  $h$  denotes the genus of  $X$ .*

*Proof.* Given a local system  $(E, \nabla)$  the first step is to push it forward along the Frobenius morphism  $\mathrm{Fr} : X \rightarrow X^{(1)}$ . This being a finite morphism we obtain  $\chi(E) = \chi(\mathrm{Fr}_* E)$ , and using Riemann-Roch for both sides we deduce the equality

$$\deg E + n(1-h) = \deg \mathrm{Fr}_* E + pn(1-h).$$

In particular we have  $\deg \mathrm{Fr}_* E = \deg E + n(1-p)(1-h)$ . If  $\pi : Y_a \rightarrow X^{(1)}$  denotes the finite morphism from the spectral curve to the base curve, we can

write  $\mathrm{Fr}_* E = \pi_*(L \otimes S)$ , where  $L$  is a line bundle on  $Y_a$ . Using again the identity  $\chi(\pi_*(L \otimes S)) = \chi(L \otimes S)$  and Riemann-Roch, we obtain

$$\deg \mathrm{Fr}_* E + pn(1 - h) = p \deg L + \deg S + p(1 - g).$$

Here we use  $g$  to denote the genus of the spectral curve. On the other hand we compute for the Higgs bundle corresponding to  $(E, \nabla)$  and  $S$ , i.e. for  $\pi_* L$  the Euler characteristic

$$\chi(\pi_* L) = \chi(L) = \deg L + (1 - g).$$

We conclude that  $p \deg \pi_* L + \deg S = \deg \pi_*(L \otimes S)$ .  $\square$

If we knew it was possible to choose  $\deg S = (1 - p)(1 - g)n$ , then we would obtain local isomorphism of moduli stacks which only multiply the degree of underlying bundles by  $p$ . But a priori it is not obvious that such a splitting even exists. The reason for this is that  $\mathcal{S}(\mathcal{D}/C_a)$  is a  $\mathcal{P}\mathrm{ic}(C_a)$ -torsor. If we tensor  $S$  with a line bundle  $L$  the degree is  $\deg L \otimes S = \deg S + p \deg L$ . Therefore  $S$  can only attain degrees in a certain congruence class of  $p$ . Let us also point out that the connected components of the moduli stack of Higgs bundles are labelled by rank and degree, therefore it suffices to compute the shift for a single geometric point.

Nonetheless this is enough to conclude that the degree of the underlying bundle of a local system on  $X$  can only attain values in a fixed congruence class of  $p$ . The trivial local system  $(\mathcal{O}_X^n, d)$  allows us to conclude that this congruence class is  $p\mathbb{Z}$ . In particular there exists a connected component of splittings  $\mathcal{S}^0$  of the desired degree.

**Lemma 3.17.** *Let  $U \rightarrow \mathcal{A}^{(1)}$  be an étale map, where  $U$  is connected. Assume there is a splitting  $S \in \mathcal{S}^0(\mathcal{D}/U)$ . Then the induced morphism  $U \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\mathrm{Dol}}(X^{(1)}) \rightarrow U \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\mathrm{dR}}(X)$  changes the degree of subbundles by multiplication with  $p$ . In particular it preserves the notion of (semi)stability.*

*Proof.* For  $n = k + l$  we consider the morphism  $\phi : \mathcal{B} := \mathcal{A}_k \times \mathcal{A}_l \rightarrow \mathcal{A}_n$  given by polynomial multiplication. Forming the base change of  $U$  we obtain a connected étale neighbourhood  $U' \rightarrow \mathcal{B}^{(1)}$  and a splitting  $S'$  of  $\mathcal{D}$  on the base change of the rank  $k$  spectral curve family. We conclude that  $S'$  multiplies the degree with  $p$  and shifts it. If the image of  $U$  contains the zero fibre, this shift can be calculated to be zero, by analyzing the local system  $(\mathcal{O}_X^n, d) = (\mathcal{O}_X, d) \oplus \dots \oplus (\mathcal{O}_X, d)$ . In general we choose an étale neighbourhood  $V \rightarrow \mathcal{A}^{(1)}$  of  $0 \in \mathcal{A}^{(1)}$  and a splitting of  $\mathcal{S}^0(V)$ . The image of  $U$  and  $V$  intersect, and for a geometric point in the intersection those shifts agree.  $\square$

A splitting from the substack  $\mathcal{S}^0$  multiplies the degree of a Higgs bundle and its subbundles by  $p$ . This finishes the proof of Theorem 3.15.

We can use this to deduce a generalization of a result of Laszlo and Pauly. In their paper [LP01] they prove that the zero fibre of the deformed Hitchin fibration from the stack of semi-stable  $t$ -connections to the Hitchin base is universally closed.

Using descent theory and properness of the Hitchin map for the moduli space of Higgs bundles (cf. [Nit91, Thm. 6.1]), this implies Corollary 1.2. Here we replace the stacks by their GIT-versions, i.e. we only consider semi-stable bundles, identify semi-stable bundles with isomorphic associated graded of the Jordan-Hölder filtration (S-equivalence) and moreover rigidify the stacks.

## 4 Langlands correspondence revisited

For a smooth proper curve  $X/\mathbb{C}$ , the geometric Langlands correspondence refers to a conjectured equivalence of categories

$$D^?( \mathcal{M}_{\mathrm{dR}}, \mathcal{O} ) \cong D^?( \mathrm{Bun}, D_{\mathrm{Bun}} ),$$

respecting various extra structures. The categories involved are expected to be certain derived categories or possibly enhancements thereof. On  $D^?( \mathrm{Bun}, D_{\mathrm{Bun}} )$  acts a family of functors, called Hecke operators. The geometric Langlands correspondence is expected to intertwine those with certain multiplication operators acting on  $D^?( \mathcal{M}_{\mathrm{dR}}, \mathcal{O} )$ .

The picture described above is reminiscent of number theory. The Langlands programme is a collection of theorems and conjectures, encompassing a far-reaching generalization of class field theory. Using Grothendieck's *function-sheaf* dictionary it is possible to relate these two programmes. We refer to the survey [Fre07] and references therein for an account of the classical and geometric Langlands conjecture.

In their paper [BB] Bezrukavnikov–Braverman could establish a derived equivalence over the locus of smooth spectral curves, relating D-modules on  $\mathrm{Bun}$  in positive characteristic to coherent sheaves on  $\mathcal{M}_{\mathrm{dR}}$ . In this section we extend their results to the locus of integral spectral curves by presenting the proof of Corollary 1.3. We construct an equivalence of derived categories  $D_{\mathrm{coh}}^b( \mathcal{M}_{\mathrm{dR}}^{\mathrm{int}}, \mathcal{O} ) \cong D_{\mathrm{coh}}^b( \mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}}, \mathcal{D}_{\mathrm{Bun}} )$ , where  $\mathcal{D}_{\mathrm{Bun}}$  denotes the Azumaya algebra of differential operators on  $\mathrm{Bun}$  defined on an open dense substack of  $\mathcal{M}_{\mathrm{Dol}}(X^{(1)})$  (see section 3.13 in [BB]). This equivalence is constructed using the Arinkin–Poincaré sheaf ([Ari10]). For this reason we assume from now on that the characteristic of the base field  $p$  satisfies the estimate

$$p > 2n^2(h-1) + 1,$$

where  $h$  denotes the genus of  $X$ . This assumption is required to apply Arinkin's results, as we will explain in Theorem 4.6 and Lemma 4.11. Moreover this equivalence can be shown to intertwine the Hecke operator with a multiplication operator.

Corollary 1.3 emphasizes the similarity of autoduality phenomena of the Hitchin system (the classical limit) with the geometric Langlands programme. It provides another example for the philosophy that *quantum* statements become *almost classical* when specialized to positive characteristic.

## 4.1 Splitting of $\mathcal{D}_{\text{Bun}}$ on smooth Hitchin fibres

This subsection is entirely expository. We begin by reviewing the theory of abelian group stacks. More details can be found in [Ari08]. In the following we fix a base scheme  $S$  and consider an abelian group  $S$ -stack  $Y$ . The dual  $Y^\vee$  is defined to be the stack classifying morphisms of group stacks  $Y \rightarrow B\mathbb{G}_m$ . It has a natural structure of a group stack. If  $A$  is an abelian  $S$ -scheme,  $A^\vee$  is the dual abelian  $S$ -scheme. Moreover we have  $B\mathbb{G}_m^\vee = \mathbb{Z}$  and  $\mathbb{Z}^\vee = B\mathbb{G}_m$ . If  $\Gamma$  is a finite group, its dual is given by the classifying stack  $B\Gamma^*$  of the Cartier dual  $\Gamma^*$ .

A group stack  $Y$  is said to be *nice*, if the natural morphism  $Y \rightarrow Y^{\vee\vee}$  is an equivalence. All the examples considered above are nice.

Dualizing is in general not an exact operation, as the following counter-example shows. An isogeny of abelian varieties  $A \rightarrow B$  gives rise to an exact sequence of nice group stacks

$$0 \rightarrow \Gamma \rightarrow A \rightarrow B \rightarrow 0,$$

the dual sequence is  $B^\vee \rightarrow A^\vee \rightarrow B\Gamma^* \rightarrow 0$ , but the first arrow is the dual isogeny and certainly has a non-trivial kernel in general.

The theory of abelian group stacks is used by Bezrukavnikov–Braverman to show the following Theorem ([BB, Thm. 4.10(1)]):

**Theorem 4.1.** *The Azumaya algebra of differential operators  $\mathcal{D}_{\text{Bun}}$  carries a natural group structure over the locus of smooth spectral curves. In particular we have an extension of group  $\mathcal{A}_{\text{sm}}^{(1)}$ -stacks*

$$0 \rightarrow B\mathbb{G}_m \rightarrow \mathcal{Y}_{\mathcal{D}_{\text{Bun}}} \rightarrow \mathcal{P}\text{ic}(Y/\mathcal{A}_{\text{sm}}^{(1)}) \rightarrow 0.$$

Here we use  $\mathcal{Y}_{\mathcal{D}_{\text{Bun}}}$  to denote the gerbe associated to  $\mathcal{D}_{\text{Bun}}$ .

We refer the reader to [OV07, App. 5.5] for a precise Definition of group structures on Azumaya algebras. For our purpose it is sufficient to know that we have an extension of group stacks as stated in the Theorem above.



Using this Theorem Bezrukavnikov–Braverman conclude (cf. [BB, Thm. 4.10(2)]) that étale locally over the Hitchin base  $\mathcal{A}_{\text{sm}}^{(1)}$  the Azumaya algebra  $\mathcal{D}_{\text{Bun}}$  splits. By virtue of the Abel-Jacobi map, they compare splittings of  $\mathcal{D}_{\text{Bun}}$  respecting the group structure on the Hitchin fibres, with splittings of  $\mathcal{D}_X$  on the spectral curve.

**Corollary 4.2.** *The Azumaya algebra  $\mathcal{D}_{\text{Bun}}$  splits étale locally over the base  $\mathcal{A}_{\text{sm}}^{(1)}$ . Moreover there exists a natural isomorphism*

$$\mathcal{S}_{\text{grp}}(\mathcal{D}_{\text{Bun}} / \mathcal{A}_{\text{sm}}^{(1)}) \cong \mathcal{S}(\mathcal{D}_X / \mathcal{A}_{\text{sm}}^{(1)}),$$

where  $\mathcal{S}_{\text{grp}}$  refers to the stack of relative splittings respecting the group structure, and we pull-back  $\mathcal{D}_X$  to the family of spectral curves.

We refer the reader again to [OV07, App. 5.5] for the notion of a splitting respecting the group structure.

## 4.2 The Langlands correspondence for $D$ -modules over the integral locus

This subsection is devoted to the proof of Theorem 4.3, which is an extension of the main result of [BB]. We use the notation  $\mathcal{M}_{\text{dR}}^{\text{int}} = \mathcal{M}_{\text{dR}}^{\text{int}}(X)$  and  $\mathcal{M}_{\text{Dol}}^{\text{int}} = \mathcal{M}_{\text{Dol}}^{\text{int}}(X^{(1)})$  for the moduli stacks of local systems on  $X$  respectively Higgs bundles on  $X^{(1)}$  of rank  $n$  with integral spectral curve.

**Theorem 4.3.** *There exists a canonical  $p_2^* \mathcal{D}_{\text{Bun}}$ -module  $\bar{\mathcal{L}}$  on*

$$\mathcal{M}_{\text{dR}}^{\text{int}} \times_{\mathcal{A}_{\text{int}}^{(1)}} \mathcal{M}_{\text{Dol}}^{\text{int}}$$

inducing an equivalence of derived categories

$$D_{\text{coh}}^b(\mathcal{M}_{\text{dR}}^{\text{int}}, \mathcal{O}) \cong D_{\text{coh}}^b(\mathcal{M}_{\text{Dol}}^{\text{int}}, \mathcal{D}_{\text{Bun}}).$$

### Functors of Fourier-Mukai type

Let us now recall the basic idea of Fourier-Mukai functors in order to clarify notation. By  $\mathcal{X}$ ,  $\mathcal{Y}$ ,  $\mathcal{Z}$ ,  $\mathcal{X} \rightarrow \mathcal{Z}$  and  $\mathcal{Y} \rightarrow \mathcal{Z}$  we denote algebraic stacks, respectively morphisms between them satisfying some mild restrictions. If  $\mathcal{P} \in D_{\text{coh}}^b(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$  we obtain an exact functor  $\Psi_{\mathcal{P}} : D_{\text{coh}}^b(\mathcal{X}) \rightarrow D_{\text{coh}}^b(\mathcal{Y})$  by stipulating

$$\mathcal{F} \mapsto Rp_{2,*}(Lp_1^* \mathcal{F} \otimes^L \mathcal{P}).$$

If  $\mathcal{W} \rightarrow \mathcal{Z}$  is another morphism of algebraic stacks and  $\mathcal{P}_1 \in D_{\text{coh}}^b(\mathcal{W} \times_{\mathcal{Z}} \mathcal{X})$ ,  $\mathcal{P}_2 \in D_{\text{coh}}^b(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$  are complexes, then we are able to define another integral kernel by convolution

$$\mathcal{P}_1 * \mathcal{P}_2 = p_{13,*}(p_{12}^* \mathcal{P}_1 \otimes p_{23}^* \mathcal{P}_2).$$

After imposing mild conditions on the stacks respectively the morphisms between them, Mukai proves the following Lemma ([Muk87]) using the base change formula.

**Lemma 4.4** (Mukai).

$$\Psi_{\mathcal{P}_2} \circ \Psi_{\mathcal{P}_1} = \Psi_{\mathcal{P}_1 * \mathcal{P}_2}.$$

**Remark 4.5.** *For a morphism*

$$\pi : \coprod_{i \in I} \mathcal{Y}_i \rightarrow \mathcal{X}$$

*we define  $\pi_* \mathcal{F}$  to be the quasi-coherent sheaf*

$$\bigoplus_{i \in I} (\pi|_{\mathcal{Y}_i})_* \mathcal{F}.$$

## Relation with the Arinkin-Poincaré sheaf

In [Ari10] Arinkin has constructed a sheaf  $\bar{\mathcal{P}}$ , extending the well-known Poincaré line bundle for Jacobian varieties of smooth curves to compactified Jacobians of integral curves with planar singularities.

**Theorem 4.6** (Arinkin). *Let  $k$  be an algebraically closed field of characteristic zero or  $p > 2g - 1$ . Let  $S$  be a  $k$ -scheme locally of finite type and  $\pi : C \rightarrow S$  a flat family of integral curves with planar singularities and arithmetic genus  $g$ . Then there exists a coherent sheaf  $\bar{\mathcal{P}}$  on  $\bar{\mathbf{J}} \times_S \bar{\mathbf{J}}$ , which is Cohen-Macaulay over every closed point  $s \in S$  and extends the Poincaré line bundle  $\mathcal{P}$  on  $\bar{\mathbf{J}} \times_S \mathbf{J} \cup \mathbf{J} \times_S \bar{\mathbf{J}}$ . Fibrewise Fourier-Mukai duality with respect to the kernel  $\bar{\mathcal{P}}$  gives rise to an equivalence of bounded derived categories*

$$\mathfrak{F} : D_{\text{coh}}^b(\bar{\mathbf{J}}) \rightarrow D_{\text{coh}}^b(\bar{\mathbf{J}}).$$

The proof of this Theorem can be found in [Ari10] for the case of characteristic zero. The reason for restricting the characteristic is that Haiman's celebrated  $n!$  Theorem (cf. [Hai01]) plays a role in the construction of  $\bar{\mathcal{P}}$ . If  $H_n$  denotes the Hilbert scheme of length  $n$  points on  $\mathbb{A}^2$ , this Theorem of Haiman shows that a certain natural morphism  $X_n \rightarrow H_n$  is finite and flat.

The scheme  $X_n$  is referred to as the *isospectral Hilbert scheme*, we conclude that it is Cohen-Macaulay. Again the original source states the relevant Theorem only in the case of characteristic zero, but the proof is easily adapted to  $p > n$ . The only characteristic-sensitive part of Haiman's proof is the use of Maschke's theorem for the symmetric group  $S_n$ , which is true as long as  $p > n$  ([Lan02, Thm. XVIII.1.2]). The combinatorial backbone of Haiman's work, the Polygraph Theorem, has already been proved over  $\mathbb{Z}$  in the original publication. In order to construct the Arinkin-Poincaré sheaf one needs Cohen-Macaulayness of the isospectral Hilbert scheme  $X_n$  for  $n = 2g - 1$ . This requires  $p > 2g - 1$ . But since representation theory of the symmetric group is also used in the defining formula [Ari10, (4.1)] of the Arinkin-Poincaré sheaf, Arinkin's methods depend on the restriction  $p > 2g - 1$  a second time.

The smooth locus of the family of compactified Jacobians  $\bar{J} \rightarrow S$  is given by the Jacobian  $J \rightarrow S$ . Over this open subscheme, the Arinkin-Poincaré sheaf  $\bar{\mathcal{P}}/\bar{J} \times \bar{J}$  restricts to the Poincaré line bundle  $\mathcal{P}/J \times J$ . Moreover if the generic member of the family  $C \rightarrow S$  is smooth, the codimension of the complement of  $J \subset \bar{J}$  is  $\geq 2$ . By the Cohen-Macaulay property this allows us to reconstruct the original sheaf by push-forward from the smooth locus

$$\bar{\mathcal{P}} = i_* \mathcal{P}.$$

The following Theorem is a direct consequence of [Ari10, Thm. C].

**Theorem 4.7.** *With the same assumptions as in Theorem 4.6 there exists a natural automorphism of derived categories*

$$\mathfrak{F} : D_{\text{coh}}^b(\overline{\mathcal{P}\text{ic}}) \rightarrow D_{\text{coh}}^b(\overline{\mathcal{P}\text{ic}})$$

given by an integral kernel  $\bar{\mathcal{P}}$ , which is a Cohen-Macaulay coherent sheaf on the stack  $\overline{\mathcal{P}\text{ic}} \times_S \overline{\mathcal{P}\text{ic}}$ . Moreover  $\bar{\mathcal{P}}$  is the push-forward of the Poincaré line bundle  $\mathcal{P}$  on  $\overline{\mathcal{P}\text{ic}} \times_S \mathcal{P}\text{ic} \cup \mathcal{P}\text{ic} \times_S \overline{\mathcal{P}\text{ic}}$ .

*Proof.* A section  $s : S \rightarrow C^{\text{sm}}$  of  $\pi : C^{\text{sm}} \rightarrow S$  gives rise to an identification

$$\overline{\mathcal{P}\text{ic}}(C/S) \cong \mathbb{Z} \times \bar{J} \times B\mathbb{G}_m,$$

respecting the group structure on  $J \subset \bar{J}$ . To see this we note first of all that  $s$  gives rise to a moduli problem for the rigidification  $\mathcal{P}\text{ic}^{\text{rig}}$ . Namely  $\mathcal{P}\text{ic}^{\text{rig}}(\tau : T \rightarrow S)$  is the set of isomorphism classes of pairs consisting of a family of line bundles and a trivialization along  $s$ :

$$(L/C \times_S T, t : (\text{id}_T \times s)^* L \cong \mathcal{O}_T).$$

Forgetting the trivialization gives rise to a morphism  $\overline{\mathcal{P}\mathrm{ic}}^{rig} \rightarrow \overline{\mathcal{P}\mathrm{ic}}$  which neutralizes the gerbe  $\overline{\mathcal{P}\mathrm{ic}} \rightarrow \overline{\mathcal{P}\mathrm{ic}}^{rig}$ , i.e. induces an identification

$$\overline{\mathcal{P}\mathrm{ic}} \cong \overline{\mathcal{P}\mathrm{ic}}^{rig} \times_B \mathbb{G}_m.$$

On the other hand, twisting by the section  $s$  gives rise to an identification

$$\mathcal{P}\mathrm{ic}^{rig} \cong \mathbb{Z} \times \bar{\mathbb{J}}.$$

Although  $\pi : C^{\mathrm{sm}} \rightarrow S$  may not allow a section globally, this is always the case locally in the étale topology.

Let  $\mathcal{Y}$  be an  $S$ -stack, with an open dense substack  $Y^0$  which is a group  $S$ -stack. We denote by  $\mathcal{Y}^\vee$  the stack classifying  $\mathbb{G}_m$ -torsors  $L$  on  $\mathcal{Y}$ , such that over every geometric point of  $S$  the restriction  $L|_{Y^0}$  can be endowed with the structure of a group stack extension of  $\mathcal{Y}^0$  and moreover the natural action of  $L|_{Y^0}$  on itself can be extended to an action of  $L|_{Y^0}$  on  $L$ .

The Abel-Jacobi morphism  $A : C \rightarrow \overline{\mathcal{P}\mathrm{ic}}$  gives rise to a morphism

$$A^* : \overline{\mathcal{P}\mathrm{ic}}^\vee \rightarrow \mathcal{P}\mathrm{ic}.$$

Étale locally we see that  $A^*$  is an isomorphism, thus it is an isomorphism globally by faithfully flat descent. This gives rise to a line bundle  $\mathcal{P} / \overline{\mathcal{P}\mathrm{ic}} \times_S \mathcal{P}\mathrm{ic}$ , which can be extended to a line bundle on

$$U = \overline{\mathcal{P}\mathrm{ic}} \times_S \mathcal{P}\mathrm{ic} \cup \mathcal{P}\mathrm{ic} \times \overline{\mathcal{P}\mathrm{ic}}$$

by symmetry. If  $j : U \rightarrow \overline{\mathcal{P}\mathrm{ic}} \times_S \overline{\mathcal{P}\mathrm{ic}}$  denotes the open inclusion of  $U$ , then we define  $\bar{\mathcal{P}}$  as  $j_* \mathcal{P}$ . Using descent theory again we see that  $\bar{\mathcal{P}}$  is a (fibrewise) Cohen-Macaulay sheaf.

Using  $\bar{\mathcal{P}}$  as integral kernel induces an automorphism of derived categories since the corresponding cohomological computations can be performed étale locally on the base  $S$ .  $\square$

## Twisting Arinkin's equivalence

We would like to construct an integral kernel, i.e. a sheaf  $\bar{\mathcal{L}}$  on the fibre product  $\mathcal{M}_{\mathrm{dR}}^{\mathrm{int}} \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}}$ , which is endowed with the structure of a  $p_2^* \mathcal{D}_{\mathrm{Bun}}$ -module. As soon as this is done, we are able to set-up a Fourier-Mukai functor between derived categories, analogous to [Muk87].

Let us denote by

$$\phi : \mathcal{S}_{\mathrm{int}}^0 \times_{\mathcal{A}_{\mathrm{int}}^{(1)}} \mathcal{M}_{\mathrm{dR}}^{\mathrm{int}} \rightarrow \mathcal{S}_{\mathrm{int}}^0 \times_{\mathcal{A}_{\mathrm{int}}^{(1)}} \mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}}$$

the isomorphism of Theorem 3.15. According to Corollary 4.2 there exists a  $p_2^* \mathcal{D}_{\text{Bun}}$ -splitting  $\mathcal{L}$  on  $\mathcal{S} \times_{\mathcal{A}_{\text{sm}}^{(1)}} \mathcal{M}_{\text{Dol}}^{\text{sm}}$ . Since the total space of  $\mathcal{S} \times_{\mathcal{A}_{\text{int}}^{(1)}} \mathcal{M}_{\text{Dol}}^{\text{int}}$  is smooth we can extend  $\mathcal{L}$  to a splitting on  $\mathcal{S} \times_{\mathcal{A}_{\text{int}}^{(1)}} \overline{\mathcal{P}\text{ic}}$ . On every connected component this extension is unique up to tensoring by a line bundle  $\chi^* L$  pulled back from  $\mathcal{A}_{\text{int}}^{(1)}$ . But  $\mathcal{P}\text{ic}(\mathcal{A}_{\text{int}}^{(1)}) = 1$ , since  $\mathcal{A}_{\text{int}}^{(1)} \subset \mathcal{A}^{(1)}$  is an open subscheme of affine space. In particular every extension is unique up to isomorphism. We choose one and denote it again by  $\mathcal{L}$ . We have  $\underline{\text{Hom}}_{p_3^* \mathcal{D}_{\text{Bun}}}(p_{13}^* \mathcal{L}, p_{23}^* \mathcal{L}) = \phi^* p_{23}^* \mathcal{P}$  on

$$\mathcal{S}_{\text{int}}^0 \times_{\mathcal{A}_{\text{int}}^{(1)}} \mathcal{M}_{\text{dR}}^{\text{int}} \times_{\mathcal{A}_{\text{int}}^{(1)}} \mathcal{M}_{\text{Dol}}^{\text{int}} \cong \mathcal{S}_{\text{int}}^0 \times_{\mathcal{A}_{\text{int}}^{(1)}} \mathcal{M}_{\text{Dol}}^{\text{int}} \times_{\mathcal{A}_{\text{int}}^{(1)}} \mathcal{M}_{\text{Dol}}^{\text{int}}.$$

We denote by

$$j : \mathcal{S} \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\text{Dol}}^{\text{int}} \rightarrow \mathcal{M}_{\text{Dol}}^{\text{int}} \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\text{Dol}}^{\text{int}}$$

the inclusion of this open substack. We observe that its complement has codimension  $\geq 2$ , since  $\chi_{dR}$  is flat and  $j$  is surjective over the generic fibre. Theorem 4.7 motivates the next definition.

**Definition 4.8.** *We define the  $p_2^* \mathcal{D}_{\text{Bun}}$ -module  $\bar{\mathcal{L}}$  to be  $j_* \mathcal{L}$ .*

Using this identification we obtain the identity

$$(\phi \times_{\mathcal{A}_{\text{int}}^{(1)}} \text{id}_{\mathcal{M}_{\text{Dol}}^{\text{int}}})^* \underline{\text{Hom}}_{p_3^* \mathcal{D}_{\text{Bun}}}(p_{13}^* \mathcal{L}, p_{23}^* \bar{\mathcal{L}}) = p_{13}^* \bar{\mathcal{P}}$$

on

$$\mathcal{S}_{\text{int}}^0 \times_{\mathcal{A}_{\text{int}}^{(1)}} \mathcal{M}_{\text{dR}}^{\text{int}} \times_{\mathcal{A}_{\text{int}}^{(1)}} \mathcal{M}_{\text{Dol}}^{\text{int}} \cong \mathcal{S}_{\text{int}}^0 \times_{\mathcal{A}_{\text{int}}^{(1)}} \mathcal{M}_{\text{Dol}}^{\text{int}} \times_{\mathcal{A}_{\text{int}}^{(1)}} \mathcal{M}_{\text{Dol}}^{\text{int}}.$$

Étale locally on  $\mathcal{A}_{\text{int}}^{(1)}$  we can choose a section of  $\mathcal{S}_{\text{int}}^0 \rightarrow \mathcal{A}_{\text{int}}^{(1)}$ . Using descent theory, we conclude the following lemmas.

**Lemma 4.9.** *The sheaf  $\bar{\mathcal{L}}$  is Cohen-Macaulay.*

**Lemma 4.10.** *The Fourier-Mukai functor*

$$\Psi_{\bar{\mathcal{L}}} : D_{\text{coh}}^b(\mathcal{M}_{\text{dR}}^{\text{int}}, \mathcal{O}) \rightarrow D_{\text{coh}}^b(\mathcal{M}_{\text{Dol}}^{\text{int}}, \mathcal{D}_{\text{Bun}})$$

*is an isomorphism.*

In order to explain the restriction we have to put on the characteristic of the base field, we need to calculate the arithmetic genus of spectral curves. From the characteristic zero theory one expects the arithmetic genus of spectral curves to be  $n^2(h-1) + 1$ , since the genus of a smooth spectral curve equals the dimension of its Picard, i.e. the corresponding Hitchin fibre. Due to the

Lagrangian property of a Hitchin fibre, this dimension is half the dimension of the total space, i.e. the same as the dimension of the moduli space of vector bundles  $n^2(h-1)+1$ . For general fields we arrive at the same number by a simple Riemann-Roch computation.

**Lemma 4.11.** *The arithmetic genus  $g$  of a spectral curve  $Y_a$  of a curve  $X$  of genus  $h$  and a Higgs bundle of rank  $n$  is given by  $g = n^2(h-1)+1$*

*Proof.* Because the arithmetic genus is constant in flat families it suffices to calculate the genus of smooth spectral curves. Let  $\pi : Y_a \rightarrow X$  denote the finite morphism of a smooth spectral curve to  $X$ . Since  $\pi$  is finite, we know that  $\chi(\pi_* \mathcal{O}_{Y_a}) = \chi(\mathcal{O}_{Y_a})$ . The right hand side is given by  $1-g$  according to the Riemann-Roch formula. The left hand side is constant in flat families and thus may be computed for a particular spectral curve. We choose  $Y_0$ , the most non-reduced spectral curve corresponding to the nilpotent cone. If  $\Theta_X$  denotes the sheaf of tangent vector fields on  $X$ , then

$$\pi_* \mathcal{O}_{Y_0} = \bigoplus_{i=0}^{n-1} \Theta_X^i.$$

Combining this with the Riemann-Roch formula we compute

$$\chi(\pi_* \mathcal{O}_{Y_0}) = \sum_{i=0}^{n-1} (1-h+2i(1-h)) = n^2(1-h).$$

In particular we obtain that the arithmetic genus of a spectral curve is given by

$$n^2(h-1)+1.$$

□

This concludes the proof of Theorem 4.3. We finish this section by an important remark, which is proved by exactly the same methods as Theorem 4.3.

**Remark 4.12.** *Let  $S$  be a smooth  $k$ -scheme locally of finite type. Then we have an equivalence of derived categories*

$$D_{coh}^b(\mathcal{M}_{\text{Dol}}^{\text{int}} \times T^*S^{(1)}, \mathcal{D}_{\text{Bun} \times S}) \cong D_{coh}^b(\mathcal{M}_{\text{dR}}^{\text{int}} \times T^*S^{(1)}, \mathcal{O}_{\mathcal{M}_{\text{dR}}} \boxtimes \mathcal{D}_S).$$

*The case  $S = X$  allows us to formulate the Hecke eigenproperty.*

### 4.3 The Hecke eigenproperty

The equivalence of Theorem 4.3 can be shown to intertwine the Hecke functor with a multiplication functor. This is expected from the Geometric Langlands conjecture over  $\mathbb{C}$ .

#### A Multiplication operator

Let  $\mathcal{E}$  denote the universal vector bundle over  $\mathcal{M}_{\text{dR}} \times X$ . It gives rise to a multiplication functor.

$$\begin{aligned} \mathbb{W} : D_{\text{coh}}^b(\mathcal{M}_{\text{dR}}, \mathcal{O}) &\rightarrow D_{\text{coh}}^b(\mathcal{M}_{\text{dR}} \times X, \mathcal{O} \boxtimes \mathcal{D}_X) \\ M &\mapsto M \otimes \mathcal{E} \end{aligned}$$

#### The Hecke operator

We define the stack  $\mathcal{H}$  to be the classifying stack of the data  $(E, F, \iota, x)$ , such that  $E, F \in \text{Bun}$ ,  $\iota : E \rightarrow F$  is an injection,  $x \in X$  and  $\text{coker } \iota$  is a skyscraper sheaf of length one.

Note that  $\mathcal{H}$  is equipped with two natural morphisms

$$\begin{array}{ccc} & \mathcal{H} & \\ q \swarrow & & \searrow p \\ \text{Bun} & & \text{Bun} \times X \end{array}$$

sending  $(E, F, \iota, x) \mapsto E$  respectively  $(E, F, \iota, x) \mapsto (F, x)$ .

**Remark 4.13.** *The stack of Hecke operators  $\mathcal{H}$  is actually a moduli stack for a certain type of (quasi-)parabolic bundles (see [BY96, Def. 2.1] for a definition). In particular we see that Serre duality for parabolic bundles as stated in Proposition 2.4 of loc. cit. allows us to think of the corresponding moduli stack of (quasi-)parabolic Higgs bundles as the cotangent stack  $T^* \mathcal{H}$ .*

We define the Hecke operator  $\mathbb{H}$  to be the functor

$$\begin{aligned} \mathbb{H} : D_{\text{coh}}^b(\text{Bun}, D) &\rightarrow D_{\text{coh}}^b(\text{Bun} \times X, D) \\ M &\mapsto p_* q^! M. \end{aligned}$$

Whereas the Definition of  $\mathbb{W}$  makes immediately sense for the smaller stack  $\mathcal{M}_{\text{dR}}^{\text{int}}$ , it is not obvious that  $\mathbb{H}$  descends to a functor

$$D_{\text{coh}}^b(\mathcal{M}_{\text{Dol}}^{\text{int}}, \mathcal{D}) \rightarrow D_{\text{coh}}^b(\mathcal{M}_{\text{Dol}}^{\text{int}} \times X, \mathcal{D}_{\text{Bun} \times X}).$$

In order to see that this is the case we need to remind the reader of the definition of the functors  $p_*$  and  $q^!$ .

### Push-forward and $!$ -pullback for D-modules in positive characteristic

We refer to [BB] for a detailed discussion of the functors  $q^!$  and  $p_*$  in the context of D-modules in positive characteristic. We just recall the basic ideas for the convenience of the reader.

Every morphism of smooth schemes  $\pi : V \rightarrow U$  induces a morphism  $d\pi : V \times_U T^*U \rightarrow T^*V$ .

On the Frobenius twist  $(V \times_U T^*U)^{(1)}$  we thus get two Azumaya algebras:  $d\pi^{(1),*} \mathcal{D}_V$  and  $p_2^* \mathcal{D}_U$ . In Proposition 3.7 of [BB] it is shown that those two Azumaya algebras are canonically equivalent, which allows us to identify the category of  $d\pi^{(1),*} \mathcal{D}_V$ -modules with the category of  $p_2^* \mathcal{D}_U$ -modules.

This yields a natural functor

$$\pi_* : D_V - \text{Mod} \rightarrow D_U - \text{Mod}.$$

If  $M \in D_V - \text{Mod}$ , we pull it back along  $d\pi^{(1)}$  and obtain a  $p_2^* \mathcal{D}_U$ -module by the natural identification cited above. Pushing it forward we get a  $\mathcal{D}_U$ -module.

Of course there is a similar description of the functor

$$\pi^! : D_U - \text{Mod} \rightarrow D_V - \text{Mod}.$$

This time we start with a  $D_U$ -module  $N$ , pull it back along  $p_2$  and obtain a  $d\pi^{(1),*} \mathcal{D}_V$ -module. By pushing it forward along  $d\pi^{(1)}$  we obtain a  $\mathcal{D}_V$ -module.

The two definitions above are easily captured in a diagram:

$$\begin{array}{ccc} p_2^* \mathcal{D}_U - \text{Mod} & \xlongequal{\quad} & d\pi^{(1),*} \mathcal{D}_V - \text{Mod} \\ p_2^* \left( \uparrow \right) p_{2,*} & & d\pi^{(1),*} \left( \uparrow \right) d\pi_*^{(1)} \\ \mathcal{D}_U - \text{Mod} & \xrightleftharpoons[\pi_*]{\pi^!} & \mathcal{D}_V - \text{Mod} \end{array}$$

### A reformulation of the Hecke operator

In order to define the functor  $q^! : \mathcal{D}_{\text{Bun}} - \text{Mod} \rightarrow \mathcal{D}_{\mathcal{H}} - \text{Mod}$  we have to consider the morphism

$$dq^{(1)} : q^{(1),*} T^* \text{Bun}^{(1)} \rightarrow T^* \mathcal{H}^{(1)},$$

and use that  $q^{(1),*} \mathcal{D}_{\text{Bun}}$  and  $dq^{(1),*} \mathcal{D}_{\mathcal{H}}$  are canonically Morita equivalent.



Analogously we need to consider

$$dp^{(1)} : p^{(1)*} T^*(\mathrm{Bun} \times X)^{(1)} \rightarrow T^* \mathcal{H}^{(1)},$$

and the Morita equivalence of  $dp^{(1)*} \mathcal{D}_{\mathcal{H}}$  with  $p^{(1)*} \mathcal{D}_{\mathrm{Bun} \times X}$  to define

$$p_* : \mathcal{D}_{\mathcal{H}} - \mathrm{Mod} \rightarrow \mathcal{D}_{\mathrm{Bun} \times X} - \mathrm{Mod}.$$

The most natural way to deal with those two morphisms simultaneously is to look at their fibre product

$$\begin{array}{ccc} Z & \xrightarrow{\quad} & q^* T^* \mathrm{Bun} \\ \downarrow & & \downarrow \\ p^* T^* \mathrm{Bun} \times X & \xrightarrow{\quad} & T^* \mathcal{H}. \end{array}$$

The stack  $Z^{(1)} = Z(X^{(1)})$  is the domain of the morphisms<sup>6</sup>

$$\alpha_1 = q \circ \mathrm{pr}_1 : Z \rightarrow \mathcal{M}_{\mathrm{Dol}}(X^{(1)})$$

and

$$\alpha_2 = p \circ \mathrm{pr}_2 : Z \rightarrow \mathcal{M}_{\mathrm{Dol}}(X^{(1)}) \times T^* X^{(1)}.$$

On  $Z$  we then have three natural Azumaya algebras,  $\alpha_1^* \mathcal{D}_{\mathrm{Bun}}$ ,  $\alpha_2^* \mathcal{D}_{\mathrm{Bun} \times X}$  and  $\pi^* \mathcal{D}_{\mathcal{H}(X^{(1)})}$ , where  $\pi : Z \rightarrow \mathcal{H}$  denotes the structural morphism of the fibre product  $Z$ . By construction, all these algebras are pairwise Morita equivalent.

Flat base change implies that

$$\mathbb{H} : M \mapsto R\alpha_{2,*} L\alpha_1^* M,$$

where the application of Morita equivalences (which involves tensoring by a splitting) has been suppressed to simplify notation. We will later turn this into a definition of  $\mathbb{H}$ .

Let us record the following observation of [BB, 4.16].

**Lemma 4.14.** *The stack  $Z$  is given by the lax 2-functor sending an  $\mathcal{A}$ -scheme  $S \rightarrow \mathcal{A}$  to the groupoid classifying  $\{x : S \rightarrow X \times \mathcal{A}, L_1 \subset L_2\}$ , such that  $L_1, L_2 \in \overline{\mathrm{Pic}}(Y/\mathcal{A}_{\mathrm{int}})$  and  $x^*(L_2/L_1)$  is of length 1.*

*Proof.* We prove this lemma on the level of  $k$ -points and leave the slightly more general case of  $S$ -families to the reader. According to remark 4.13 the stack  $T^* \mathcal{H}$  classifies the data

$$(E, F, \theta, x, \xi),$$

---

<sup>6</sup>Note that we use the notation  $p$  and  $q$  to denote morphisms which are really base changes thereof.

where  $(x, \xi) \in T^*X^{(1)}$ ,  $(E, F, x) \in \mathcal{H}$  and

$$\theta : F \rightarrow F \otimes \Omega_{X^{(1)}}^1,$$

such that  $\text{res } \theta$  is a nilpotent endomorphism of the fibre  $F \otimes k_x$  factoring through a linear map

$$F/E \rightarrow E.$$

The morphism

$$T^*(\text{Bun} \times X) \times_{\text{Bun} \times X} \mathcal{H} \rightarrow T^* \mathcal{H}$$

sends

$$[(F, \theta, x, \xi), (E, F, x)] \mapsto (E, F, \theta, x, \xi).$$

Note that  $\text{res } \theta = 0$  in this particular case. The morphism

$$T^* \text{Bun} \times_{\text{Bun}} \mathcal{H} \rightarrow T^* \mathcal{H}$$

is given by

$$[(E, \theta), (E, F)] \mapsto (E, F, \theta', x, \xi),$$

where we use that  $E|_{X-\{x\}} \cong F|_{X-\{x\}}$  and therefore the Higgs field  $\theta$  on  $E$  induces a Higgs field  $\theta'$  with a simple pole at  $x$  on  $F$ . By construction this is a (quasi-)parabolic Higgs bundle. The 1-form  $\xi$  at  $x$  is the eigenvalue of the Higgs field  $\theta'$  on the length one quotient  $F/E$ . Note that this is a sensible definition since  $\theta'$  preserves  $E$  by construction, and that  $\text{res}(\theta) : E/F \rightarrow E/F$  is the zero map according to the axioms of parabolic Higgs bundles.

Computing the base change  $Z$  now, with this information at hand, we obtain that  $Z$  classifies

$$(E, F, \theta, x, \xi),$$

where  $(F, \theta)$  is a Higgs bundle,  $E \subset F$  is preserved by  $\theta$  and  $F/E$  is a length one sheaf acted on by  $\theta$  with eigenvalue  $\xi$ .  $\square$

Finally we obtain a definition of  $\mathbb{H}$  which can be used in our context. We observe that the two morphisms to the Hitchin base  $Z \rightarrow \mathcal{A}$ , given by  $\chi \circ \alpha_1$  and  $\chi \circ \text{pr}_1 \circ \alpha_2$  agree. This is a consequence of Lemma 4.14, as a point of  $Z$  consists of two Higgs bundles identified away from a point  $x$ . In particular they have the same characteristic polynomial. This allows us to view  $Z^{(1)}$  as  $\mathcal{A}^{(1)}$ -stack, and in particular to form the base change over the integral locus. From now on all  $\mathcal{A}^{(1)}$ -stacks are understood to be restricted to  $\mathcal{A}_{\text{int}}^{(1)}$ . This will simply be omitted from the notation. Using  $Z^{(1)}$  as a correspondence, we obtain a functor

$$\mathbb{H} : D_{\text{coh}}^b(\mathcal{M}_{\text{Dol}}^{\text{int}}, \mathcal{D}_{\text{Bun}}) \rightarrow D_{\text{coh}}^b(\mathcal{M}_{\text{Dol}}^{\text{int}} \times T^*X^{(1)}, \mathcal{D}_{\text{Bun} \times X}).$$

Interpreting the category on the left-handside as a derived category of  $D$ -modules on  $\text{Bun}$  supported on the integral locus, and the right-handside as an analogous category of  $D$ -modules on  $\text{Bun} \times X$ , we can be satisfied with  $\mathbb{H}$  as a positive characteristic analogue of Hecke functors. The remainder of this paper is devoted to the proof of the following theorem, which is a formal consequence of Theorem 4.16 below.

**Theorem 4.15.** *The equivalence of Theorem 4.3 intertwines  $\mathbb{H}$  with  $\mathbb{W}$ , i.e. it gives rise to the following 2-commutative diagram of derived categories*

$$\begin{array}{ccc} D_{\text{coh}}^b(\mathcal{M}_{\text{dR}}^{\text{int}}, \mathcal{O}) & \longrightarrow & D_{\text{coh}}^b(\mathcal{M}_{\text{dR}}^{\text{int}} \times T^*X^{(1)}, \mathcal{O}_{\mathcal{M}_{\text{dR}}} \boxtimes \mathcal{D}_X) \\ \downarrow & & \downarrow \\ D_{\text{coh}}^b(\mathcal{M}_{\text{Dol}}^{\text{int}}, \mathcal{D}_{\text{Bun}}) & \longrightarrow & D_{\text{coh}}^b(\mathcal{M}_{\text{Dol}}^{\text{int}} \times T^*X^{(1)}, \mathcal{D}_{\text{Bun} \times X}). \end{array}$$

### The Hecke eigenproperty of the Arinkin-Poincaré sheaf

In order to establish that the equivalence of Theorem 4.3 intertwines the Hecke operator  $\mathbb{H}$  with a multiplication operator, we will show that the Arinkin-Poincaré sheaf  $\bar{\mathcal{P}}$  satisfies a similar property. Given a  $k$ -scheme  $S$  of finite type and a flat family of integral curves with planar singularities  $C \rightarrow S$  we can construct the compactified relative Picard stack  $\overline{\mathcal{P}\text{ic}} \rightarrow S$ , which classifies flat families of rank one torsion free sheaves  $L$  on  $C/S$ . Let us denote the universal family on  $\overline{\mathcal{P}\text{ic}} \times_S C$  by  $\bar{\mathcal{Q}}$ . The fibre product  $\overline{\mathcal{P}\text{ic}} \times_S \overline{\mathcal{P}\text{ic}}$  is endowed with a Cohen-Macaulay sheaf  $\bar{\mathcal{P}}$ , which induces an equivalence  $\mathfrak{F}_{C/S} : D_{\text{coh}}^b(\overline{\mathcal{P}\text{ic}}) \rightarrow D_{\text{coh}}^b(\overline{\mathcal{P}\text{ic}} \times_S C)$  ([Ari10, Thm. C]).

Moreover we have a space of Hecke alterations  $\mathcal{H}$  classifying quadruples  $(L_1, L_2, \iota, x)$ , such that  $L_i \in \overline{\mathcal{P}\text{ic}}$  and  $\iota : L_1 \subset L_2$  and  $\text{coker } \iota$  is a length one coherent sheaf supported at  $x$ . Note that we have a natural morphism  $\mathcal{H} \rightarrow \overline{\mathcal{P}\text{ic}} \times \overline{\mathcal{P}\text{ic}} \times C$  and composing it with the projections  $p_1$  and  $p_{23}$  we obtain morphisms  $q : \mathcal{H} \rightarrow \overline{\mathcal{P}\text{ic}}$  and  $p : \mathcal{H} \rightarrow \overline{\mathcal{P}\text{ic}} \times_S C$ . We can now define the Hecke functor

$$\mathbb{H}_{C/S} = Rp_* \circ Lq^* : D_{\text{coh}}^b(\overline{\mathcal{P}\text{ic}}) \rightarrow D_{\text{coh}}^b(\overline{\mathcal{P}\text{ic}} \times_S C).$$

The following Theorem has been stated in [Ari10, sect. 6.4].

**Theorem 4.16** (Arinkin). *The Fourier-Mukai transform  $\mathfrak{F}$  intertwines the Hecke functor  $\mathbb{H}_{C/S}$  with the multiplication functor  $- \boxtimes^L \bar{\mathcal{Q}}$ . In other words,*

we have a 2-commutative diagram of categories:

$$\begin{array}{ccc}
D_{\text{coh}}^b(\overline{\mathcal{P}\text{ic}}) & \xrightarrow{-\boxtimes^L \bar{\mathcal{Q}}} & D_{\text{coh}}^b(\overline{\mathcal{P}\text{ic}} \times_S C) \\
\mathfrak{F}_{C/S} \downarrow & & \downarrow \mathfrak{F}_{C \times_S C/C} \\
D_{\text{coh}}^b(\overline{\mathcal{P}\text{ic}}) & \xrightarrow{\mathbb{H}_{C/S}} & D_{\text{coh}}^b(\overline{\mathcal{P}\text{ic}} \times_S C)
\end{array}$$

The proof of this Theorem follows the same strategy as [Ari10]. We replace  $S$  by the moduli stack  $\mathcal{M}_g$  of integral curves with planar singularities of arithmetic genus  $g$  and study the universal family  $\mathcal{C} \rightarrow \mathcal{M}_g$ . In this case  $\overline{\mathcal{P}\text{ic}}$  is a smooth stack (cf. [FGvS99, Thm B.2]).

Translating Theorem 4.16 to integral kernels, we obtain the following equivalent statement.

**Proposition 4.17.** *We have*

$$R(p \times_{\mathcal{M}_g} \text{id}_{\overline{\mathcal{P}\text{ic}}})_* L(q \times_{\mathcal{M}_g} \text{id}_{\overline{\mathcal{P}\text{ic}}})^* \bar{\mathcal{P}} = \bar{\mathcal{P}} \boxtimes^L \bar{\mathcal{Q}}.$$

We denote by  $\mathcal{H}^{\text{sm}}$  the open substack of  $\mathcal{H}$  given by the preimage  $p^{-1}(\mathcal{P}\text{ic} \times C)$ . By definition  $\mathcal{H}^{\text{sm}}$  classifies quadruples  $(L_1, L_2, \iota, x)$ , where  $L_2$  is a line bundle. Note that  $L_1$  is then uniquely defined through  $L_2$  and  $x$ . It is given by  $\mathcal{I}_x \otimes L_2$ , i.e. the twist of the ideal sheaf of  $x$  by the line bundle  $L_2$ . This implies the following Lemma.

**Lemma 4.18.** *The restriction  $p|_{\mathcal{H}^{\text{sm}}}$  is an isomorphism onto its image.*

Let us now restrict Proposition 4.17 to the open substack  $\mathcal{P}\text{ic} \times_{\mathcal{M}_g} \overline{\mathcal{P}\text{ic}} \times_{\mathcal{M}_g} C$ . The Hecke functor on the left-hand-side may be replaced by  $L(q|_{\mathcal{H}^{\text{sm}} \times \text{id}_{\overline{\mathcal{P}\text{ic}}}})^*$ , since  $p$  is an isomorphism. Both sides of the identity are underived, as  $\mathcal{P}$  is an invertible sheaf.

**Lemma 4.19.** *Let  $q : \mathcal{P}\text{ic} \times_{\mathcal{M}_g} C \rightarrow \overline{\mathcal{P}\text{ic}}$  be the morphism sending a pair  $(L, x)$  consisting of a line bundle  $L$  on  $C$  and a point  $x \in C$  to the twist  $L(-x) = \mathcal{I}_x \otimes L$ . Then we have*

$$(q \times_{\mathcal{M}_g} \text{id}_{\overline{\mathcal{P}\text{ic}}})^* \bar{\mathcal{P}} = \bar{\mathcal{P}} \boxtimes \bar{\mathcal{Q}}.$$

*In particular Proposition 4.17 holds when restricted to  $\mathcal{P}\text{ic} \times_{\mathcal{M}_g} \overline{\mathcal{P}\text{ic}}$ .*

*Proof.* Due to the Cohen-Macaulay property of  $\mathcal{P}$  and the fact that

$$\mathcal{P}\text{ic} \times_{\mathcal{M}_g} \mathcal{P}\text{ic} \times_{\mathcal{M}_g} C^{\text{sm}} \subset \mathcal{P}\text{ic} \times_{\mathcal{M}_g} \overline{\mathcal{P}\text{ic}} \times_{\mathcal{M}_g} C$$

has a complement of codimension 2, it suffices to check the identity restricted to  $\mathcal{P}ic \times_{\mathcal{M}_g} \mathcal{P}ic \times_{\mathcal{M}_g} \mathcal{C}^{\text{sm}}$ . Over this locus we are able to describe  $\mathcal{P} |_{\mathcal{P}ic \times_{\mathcal{M}_g} \mathcal{P}ic}$  as a family of  $\mathbb{G}_m$ -extensions of  $\mathcal{P}ic$ . In particular, if

$$(m, id) : \mathcal{P}ic \times_{\mathcal{M}_g} \mathcal{P}ic \times_{\mathcal{M}_g} \mathcal{P}ic \rightarrow \mathcal{P}ic \times_{\mathcal{M}_g} \mathcal{P}ic$$

is given by identity in the second component and multiplication in the first, we have an isomorphism

$$(m, id)^* \mathcal{P} \cong p_{13}^* \mathcal{P} \otimes p_{23}^* \mathcal{P}.$$

If  $A : \mathcal{C}^{\text{sm}} \rightarrow \mathcal{P}ic$  denotes the Abel-Jacobi map  $x \mapsto \mathcal{I}_x$  then  $q = m \circ (A, id)$  and

$$A^*(\mathcal{P} |_{\mathcal{P}ic \times_{\mathcal{M}_g} \mathcal{P}ic}) \cong \mathcal{Q} |_{\mathcal{P}ic \times_{\mathcal{M}_g} \mathcal{C}}.$$

In particular we obtain the required isomorphism via base change.  $\square$

**Lemma 4.20.** *For every  $C \in \mathcal{M}_g$  there exists a versal deformation  $\mathcal{C}'$  along a complete local ring  $U = \text{Spec } \hat{R} \rightarrow \mathcal{M}_g$ , such that  $U \times_{\mathcal{M}_g} \mathcal{C}$  has a Zariski covering  $\{V_i\}_{i \in I}$  and for every  $i$  we have an open immersion  $V_i \rightarrow U \times \mathbb{A}^2$ .*

*Proof.* This is a combination of Lemma A.2 and Proposition A.3 in [FGvS99].  $\square$

**Lemma 4.21.** *The morphism  $\mathcal{H} \rightarrow \mathcal{M}_g$  is syntomic<sup>7</sup> of relative dimension  $g + 1$ . Moreover it is fibrewise irreducible.*

*Proof.* For  $d \geq 2g - 1$  the morphism  $\text{Hilb}^d(\mathcal{C}) \rightarrow \overline{\mathcal{P}ic}^d(\mathcal{C})$  is smooth and faithfully flat. In this picture  $\mathcal{H}$  corresponds to  $\text{Hilb}^{n+1,n}(\mathcal{C})$ . We will show that the latter is a locally complete intersection. As we have seen in Lemma 4.20 we can pick a versal deformation  $\mathcal{C}' \rightarrow U$  of  $C \in \mathcal{M}_g$  together with a Zariski covering  $\bigcup_{i \in I} V_i = \mathcal{C}'$  and open immersions  $V_i \rightarrow U \times \mathbb{A}^2$ . Therefore we obtain a cartesian square

$$\begin{array}{ccc} \text{Hilb}^{d+1,d}(V_i/U) & \longrightarrow & \text{Hilb}^{d+1,d}(U \times \mathbb{A}^2/U) \\ \downarrow & & \downarrow \\ \text{Hilb}^{d+1}(V_i/U) & \longrightarrow & \text{Hilb}^{d+1}(U \times \mathbb{A}^2/U). \end{array}$$

From this it is easy to conclude that the relative dimension of  $\mathcal{H}$  is  $g + 1$ . We only need to show that for an integral curve  $C$  of arithmetic genus  $g$  and planar singularities,  $\dim H^{n+1,n}(C) = n + 1$ . This can be done as in the

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<sup>7</sup>i.e. locally of finite presentation, flat and fibrewise of locally complete intersection

proof of Theorem 5 in [AIK77], complementing the estimate (5.1) by the same inequality for nested Hilbert schemes.

It is known (cf. [AIK77, Cor. 7]) that  $\mathrm{Hilb}^{d+1}(\mathcal{C}'/U) \subset \mathrm{Hilb}^{d+1}(U \times \mathbb{A}^2/U)$  is a locally complete intersection. Since its base change  $\mathrm{Hilb}^{d+1,d}(\mathcal{C}'/U)$  has the same codimension in  $\mathrm{Hilb}^{d+1,d}(U \times \mathbb{A}^2/U)$ , and the latter stack is smooth<sup>8</sup>, we have shown that the total space  $\mathcal{H}$  is a locally complete intersection, in particular it is Cohen-Macaulay. Because  $\mathcal{M}_g$  is smooth ([Ari07, Prop. 4]) we see that  $\mathcal{H} \rightarrow \mathcal{M}_g$  is flat, since the dimension of the fibres is constant. Applying the above argument fibrewise, we see that the fibres are locally complete intersections. Irreducibility is verified as in the proof of Theorem 5 in [AIK77].  $\square$

**Lemma 4.22.** *The complex  $L(q \times \mathrm{id})^* \bar{\mathcal{P}}$  is a Cohen-Macaulay sheaf  $(q \times \mathrm{id})^* \bar{\mathcal{P}}$ .*

*Proof.* This follows from Lemma 2.3 in [Ari10], if we can show that  $q$  is Tor-finite, as we already know from Lemma 4.21 that  $\mathcal{H} \rightarrow \mathcal{M}_g$  is a syntomic morphism. Note that a flat base change of a syntomic (respectively Tor-finite) morphism is again syntomic (respectively Tor-finite). According to Theorem B.2 in [FGvS99] the total space of  $\bar{\mathcal{P}}\mathrm{ic}$  is smooth. For this reason every morphism mapping into  $\bar{\mathcal{P}}\mathrm{ic}$  is Tor-finite.  $\square$

Now we have almost all ingredients ready to prove Proposition 4.17. The final ingredient is the following criteria for Cohen-Macaulyness ([Ari10, Lemma 7.7]):

**Lemma 4.23.** *Let  $U$  be a scheme of pure dimension and  $M \in D_{\mathrm{coh}}^b(U)$ . If  $\mathrm{codim} \mathrm{supp} M \geq d$ ,  $M \in D^{\leq 0}(U)$ , and  $\mathbb{D}M \in D^{\leq d}(U)$ , then  $M$  is a Cohen-Macaulay sheaf of codimension  $d$ .<sup>9</sup>*

We intend to apply this Lemma to the integral kernel  $\Theta$  of the Fourier-Mukai transform  $\Psi_{\bar{\mathcal{P}}^\vee} \circ \mathbb{H}_{C/\mathcal{M}_g} \circ \Psi_{\bar{\mathcal{P}} \boxtimes \mathcal{O}_C}$ . If we can show that it is a Cohen-Macaulay sheaf of the right codimension, then it suffices to determine it in a complement of a closed subset (of the support) of codimension 2.

**Lemma 4.24.** *We have  $\mathrm{codim} \mathrm{supp} \Theta \geq g$  and every maximal-dimensional component intersects  $\Delta_{\bar{\mathcal{P}}\mathrm{ic}/\mathcal{M}_g} \times_{\mathcal{M}_g} \mathcal{C}$ .*

*Proof.* Let  $(F_1, F_2, x) \in \mathrm{supp} \Theta$ . By base change this is the case if and only if there exists  $i \in \mathbb{Z}$ , such that  $\mathbb{H}^i(\mathcal{H}, \mathbb{H}(\bar{\mathcal{P}}_{F_1}) \otimes \bar{\mathcal{P}}_{F_2}^\vee) \neq 0$ . As in Proposition 7.2 of [Ari10] we claim that  $\mathbb{H}(\bar{\mathcal{P}}_{F_1}) \otimes \bar{\mathcal{P}}_{F_2}^\vee$  is  $T$ -equivariant, where  $T$  denotes the  $\mathbb{G}_m$ -extension of  $\mathcal{P}\mathrm{ic}$  associated to  $\mathcal{P}_1 \otimes \mathcal{P}_2^\vee$ .

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<sup>8</sup>This is proved in [Che98], see section 0.2 for a statement of his result. Although the author assumes  $k = \mathbb{C}$  this statement and its proof are true for general algebraically closed fields.

<sup>9</sup>We use the convention that the dualizing complex  $\omega_U = \mathbb{D}\mathcal{O}_U$  is concentrated in degree 0.

We denote by  $T_i$  the  $\mathbb{G}_m$ -extension of  $\mathcal{P}\mathrm{ic}$  associated to  $\bar{\mathcal{P}}_{F_i}$ . The sheaf  $\bar{\mathcal{P}}_{F_i}$  has a natural  $T_1$ -equivariant structure ([Ari10, Lemma 6.5]). The morphisms  $p$  and  $q$  in the correspondence diagram defining  $\mathbb{H}$  are  $T_1$ -equivariant. Consequently  $\mathbb{H}(\bar{\mathcal{P}}_{F_1})$  is an element of the  $T_1$ -equivariant derived category.

For this reason the tensor product  $\mathbb{H}(\bar{\mathcal{P}}_{F_1}) \otimes \bar{\mathcal{P}}_{F_2}^\vee$  lies in the  $T$ -equivariant derived category.

Consequently the hypercohomology group  $\mathbb{H}^i(\mathcal{H}, \mathbb{H}(\bar{\mathcal{P}}_{F_1}) \otimes \bar{\mathcal{P}}_{F_2}^\vee) \neq 0$  carries a  $T$ -action, such that the  $\mathbb{G}_m$ -part acts tautologically. If this group was non-zero, there would be a one-dimensional  $T$ -invariant subspace, as  $T$  is abelian. This would provide a splitting of the extension  $0 \rightarrow \mathbb{G}_m \rightarrow T \rightarrow \mathcal{P}\mathrm{ic} \rightarrow 0$ . We conclude that  $F_1|_{C^{\mathrm{sm}}} = F_2|_{C^{\mathrm{sm}}}$  by pulling back along the Abel-Jacobi map. If  $\tilde{g}$  denotes the genus of the normalization of  $C$ , then the dimension of the subspace of pairs of torsion free sheaves of rank 1 satisfying  $F_1|_{C^{\mathrm{sm}}} = F_2|_{C^{\mathrm{sm}}}$  is  $2g - \tilde{g}$ . But by Proposition 6 in [Ari07] the strata  $\mathcal{M}^{\tilde{g}}$  of curves of geometric genus  $\tilde{g}$  has codimension  $\geq g - \tilde{g}$ . This proves the first part of the claim.

To prove the second assertion it suffices to note that Lemma 4.19 implies

$$\Theta \cap \mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{C}^{\mathrm{sm}} = \Delta_{\mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{C}^{\mathrm{sm}}}.$$

□

**Lemma 4.25.** *We have  $\Theta \in D^{\leq g}(\mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{C})$ .*

*Proof.* We denote by  $\mathbb{H}\bar{\mathcal{P}}$  the complex

$$R(p \times_{\mathcal{M}_g} \mathrm{id}_{\bar{\mathcal{P}\mathrm{ic}}})_* L(q \times_{\mathcal{M}_g} \mathrm{id}_{\bar{\mathcal{P}\mathrm{ic}}})^* \bar{\mathcal{P}} \in D_{\mathrm{coh}}^b(\bar{\mathcal{P}\mathrm{ic}} \times_{\mathcal{M}_g} \bar{\mathcal{P}\mathrm{ic}} \times_{\mathcal{M}_g} \mathcal{C}).$$

As seen in Lemma 4.21 the morphism  $\mathcal{H} \rightarrow \mathcal{M}_g$  is fibrewise irreducible and of dimension  $g + 1$ . In particular we conclude that  $\mathrm{supp} H^i(\mathbb{H}\bar{\mathcal{P}})$  is of relative dimension  $\leq g - i$  over the parametrizing component  $\bar{\mathcal{P}\mathrm{ic}}$ .

The integral kernel  $\Theta$  is given by convolution

$$\bar{\mathcal{P}}^\vee * \mathbb{H}\bar{\mathcal{P}} = Rp_{13,*}(Lp_{12}^* \bar{\mathcal{P}}^\vee \otimes^L Lp_{23}^* \mathbb{H}\bar{\mathcal{P}}).$$

The dimension estimate above implies that  $H^i(\Theta) = 0$  if  $i > g$ . □

*Proof of Theorem 4.16.* We apply Lemma 4.23 to the integral kernel  $\Theta[g]$ . We have already checked two of the three necessary conditions in Lemma 4.24 and 4.25. Moreover we know that the Theorem is true when restricted to the complement of a codimension two subvariety (cf. Lemma 4.19). Therefore it suffices to check the last condition of Arinkin's criteria for Cohen-Macaulay sheaves. We need to show that  $H^i(\mathbb{D}\Theta[g]) = 0$  if  $i > g$ . From Grothendieck-Serre duality it follows that

$$\mathbb{D}\Theta = \mathbb{D}Rp_{13,*}(Lp_{12}^* \bar{\mathcal{P}}^\vee \otimes^L Lp_{23}^* \mathbb{H}\bar{\mathcal{P}}) = (Rp_{13,*} \mathbb{D}(Lp_{12}^* \bar{\mathcal{P}}^\vee \otimes^L Lp_{23}^* \mathbb{H}\bar{\mathcal{P}}))[g],$$

which in turn can be simplified as

$$Rp_{13,*}\mathbb{D}(Lp_{12}^*\bar{\mathcal{P}}^\vee \otimes^L Lp_{23}^*\mathbb{H}\bar{\mathcal{P}}) = Rp_{13,*}[\omega_{\mathcal{P}\mathrm{ic}^3}p_{23}^*\omega_{\mathcal{P}\mathrm{ic}^2}^{-1}Lp_{12}^*\bar{\mathcal{P}} \otimes^L (Lp_{23}^*\mathbb{D}\mathbb{H}\bar{\mathcal{P}})].$$

Using Grothendieck-Serre duality again we see that

$$\mathbb{D}\mathbb{H}\bar{\mathcal{P}} = (p \times \mathrm{id})_*\mathbb{D}(q \times \mathrm{id})^*\bar{\mathcal{P}}.$$

According to Lemma 4.22 the sheaf  $(q \times \mathrm{id})^*\bar{\mathcal{P}}$  is Cohen-Macaulay, therefore  $\mathbb{D}(q \times \mathrm{id})^*\bar{\mathcal{P}}$  is a sheaf itself. Applying the same reasoning as in Lemma 4.25 we see that  $\mathbb{D}\Theta \in D^{\leq 0}(\overline{\mathcal{P}\mathrm{ic}} \times_{\mathcal{M}_g} \overline{\mathcal{P}\mathrm{ic}} \times_{\mathcal{M}_g} \mathcal{C})$ . We conclude that  $\Theta[g]$  is a Cohen-Macaulay sheaf.  $\square$

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